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Some Common Fixed Point Theorems for Three Mappings in Cone Metric Spaces

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Abstract—*In this paper, we prove the existence of coincidence points and common fixed point theorems in cone metric spaces for three mappings satisfying contractive conditions without exploiting the notation of continuity of any map involved therein.*

1. INTRODUCTION

In 2007, Huang and Zhang [1] generalized the concept of a metric space, replacing the set of real numbers by ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Subsequently, the study of fixed point theorems in such spaces is followed by some other mathematicians; see [2-8].

Consistent with Huang and Zhang [1], the following definitions and results will be needed.

Definition 1.1.[1] Let E be a real Banach space. A subset P of E is called a cone if and only if

- P is closed, nonempty and $P \neq \{0\}$;
- $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + bx \in P$;
- $x \in P$and $-x \in P \Rightarrow x = 0$.

Definition 1.2.[1]Let X be a nonempty set. Suppose that $d : X \times X \to E$ satisfies

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

The concept of cone metric space is more general than that of a metric space.

Definition 1.3.[1] Let E be a Banach space and $P \subset E$ be an order cone. The order cone P be called normal if there exists $L > 0$ such that for all $x, y \in E, 0 \leq x \leq y$implies $\| x \| \leq L \| y \|$.

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4.[1]Let P be a cone in Banach space E define partial ordering $\leq$ with respect to p by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate$x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of the set P. This cone P is called an order cone.

Definition 1.5.[1]Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(e) a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there is an N such that $n, m > N$, $d(x_n, x_m) \ll c$;
(f) a convergent sequence if for every $c \in E$ with $0 \ll c$, there is an N such that for all $n > N, d(x_n, x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$as $n \to \infty$. It is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

Definition 1.6.[1]Let $f, g : X \to X$. Then the pair $(f, g)$is said to be (IT)-Commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

2. MAIN RESULTS

Theorem 2.1. Let (X, d) be a metric space, and P be a normal cone with normal constant L. Suppose mappings $f, g, h : X \to X$ satisfy the contractive condition :

$$d(fx, gy) \leq \alpha d(hx, gy) + \beta d(fx, fy)$$
$$+\gamma d(fx, gy) + \delta d(fx, hx)$$
$$+\mu d(gy, hy) \qquad (2.1.1)$$

Where $\alpha, \beta, \gamma, \delta, \mu > 0and2\alpha + \beta + \gamma + \delta + \mu < 1$. If $f(X) \cup g(X) \subset h(X)$ is complete subspace of X. Then the maps f, g and h have a coincidence point $z$in X. Moreover if (f, h) and (g, h) are (IT)-Commutative at $z$then f, g and h have unique common fixed point.

Proof. Let x_0 be an arbitrary point of X, and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = hx_{2n+1} \text{and}$$

$$y_{2n+1} = gx_{2n+1} = hx_{2n+2},$$

for all $n = 0, 1, 2, \ldots.$. by equation (1), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\ &\le \alpha d(hx_{2n}, gx_{2n+1}) + \beta d(fx_{2n}, fx_{2n+1}) \\ &\quad + \gamma d(fx_{2n}, gx_{2n+1}) + \delta d(fx_{2n}, hx_{2n}) \\ &\quad + \mu d(gx_{2n+1}, hx_{2n+1}) \end{aligned}$$

$$= \alpha d(y_{2n-1}, y_{2n+1}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(y_{2n}, y_{2n+1}) + \delta d(y_{2n}, y_{2n-1}) + \mu d(y_{2n+1}, y_{2n})$$

$$\le \alpha\{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\} + \beta d(y_{2n}, y_{2n+1}) + \gamma d(y_{2n}, y_{2n+1}) + \delta d(y_{2n}, y_{2n-1}) + \mu d(y_{2n+1}, y_{2n})$$

$$(1 - \alpha - \beta - \gamma - \mu) d(y_{2n}, y_{2n+1}) \le (\alpha + \delta) d(y_{2n}, y_{2n-1})$$

$$d(y_{2n}, y_{2n+1}) \le \frac{\alpha + \delta}{1 - \alpha - \beta - \gamma - \mu} d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \le k d(y_{2n-1}, y_{2n})$$

Where $k = \frac{\alpha+\delta}{1-\alpha-\beta-\gamma-\mu}$.

Similarly, it can be shown that

$$d(y_{2n+1}, y_{2n+2}) \le d(y_{2n}, y_{2n+1})$$

Therefore, for all n,

$$d(y_{n+1}, y_{n+2}) \le k d(y_n, y_{n+1}) \le \cdots \le k^{n+1} d(y_0, y_1)$$

Now, for any $m > n$,

$$d(y_n, y_m) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m)$$

$$\le [k^n + k^{n+1} + \cdots + k^{m-1}) d(y_1, y_0)$$

$$\le \frac{k^n}{1-k} d(y_1, y_0)$$

From (1.3), we have

$\| d(y_n, y_m) \| \le \frac{k^n}{1-k} L \| d(y_1, y_0) \|$.

Which implies that $d(y_n, y_m) \to 0$ as

$n, m \to \infty$, (since $k < 1$). Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{hx_n\}$.

Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete, there exists z in $h(X)$ such that $hx_n \to z$as $n \to \infty$. Consequently, we can find z in X such that $h(z) = s$. We shall show that $hz = fz = gz$. Consider,

$$d(fz, gx_{2n+1}) \le \alpha d(hz, gx_{2n+1}) + \beta d(fz, fx_{2n+1}) + \gamma d(fz, gx_{2n+1}) + \delta d(fz, hz) + \mu d(gx_{2n+1}, hx_{2n+1})$$

Letting $n \to \infty$, we get

$$d(fz, s) \le \alpha d(s, s) + \beta d(fz, s) + \gamma d(fz, s) + \delta d(fz, s) + \mu d(s, s)$$

$d(fz, s) \le (\beta + \gamma + \delta) d(fz, s)$, a contradiction

Therefore

$$fz = s = hz. \tag{2.1.2}$$

similarly

$$d(gz, fx_{2n}) \le \alpha d(hz, fx_{2n}) + \beta d(gz, gx_{2n}) + \gamma d(gz, fx_{2n}) + \delta d(gz, hz) + \mu d(fx_{2n}, hx_{2n})$$

$$d(gz, s) \le \alpha d(s, s) + \beta d(gz, s) + \gamma d(gz, s) + \delta d(gz, s) + \mu d(s, s)$$

$d(gz, s) \le (\beta + \gamma + \delta) d(gz, s)$, a contradiction.

Therefore,

$$gz = s = hz. \tag{2.1.3}$$

From (2.1.2) and (2.1.3), it follows that

$hz = fz = gz = s$, z is a coincidence point f, g, h.(2.1.4)

Since $(f, h), (g, h)$ are (IT)-commuting at z.

$$\begin{aligned} d(ffz, fz) &= d(ffz, gz) \\ &\le \alpha d(hfz, gz) + \beta d(ffz, fz) \\ &\quad + \gamma d(ffz, gz) + \delta d(ffz, hfz) \\ &\quad + \mu d(gz, hz) \end{aligned}$$

$$\le \alpha d(fhz, gz) + \beta d(ffz, fz) + \gamma d(ffz, gz) + \delta d(ffz, fhz) + \mu d(gz, hz)$$

$$\le \alpha d(fz, gz) + \beta d(ffz, fz) + \gamma d(ffz, gz) + \delta d(ffz, fz) + \mu d(gz, hz)$$

$$< (\beta + \gamma + \delta) d(ffz, fz)$$

a contradiction, since $k < 1$ and $fz = hz$, which implies $ffz = fz$.

$$fz = ffz = fhz = hfz,$$

$$\Rightarrow ffz = hfz = fz = s. \tag{2.1.5}$$

Therefore

$fz = s$is a common fixed point of g and h.(2.1.6)

Similarly, we get,

$$ggz = ggz = ghz = hgz,$$

$$\Rightarrow ggz = hgz = gz = s. \tag{2.1.7}$$

Therefore

$gz = fz = s$is a common fixed point of g and h. (2.1.8)

In view of (2.1.6) and (2.1.8), it follows that f, g and h have a common fixed point namely s. The uniqueness of the common fixed point of s follows equation (2.1.1). Indeed, let s_1 is another common fixed point of $f, g$and h. Consider,

$$d(s,s_1) = d(fs,gs_1) \leq \alpha d(hs,gs_1) + \beta d(fs,fs_1) + \gamma d(fs,gs_1) + \delta d(fs,hs) + \mu d(gs_1,hs_1)$$

$$= \alpha d(s,s_1) + \beta d(s,s_1) + \gamma d(s,s_1) + \delta d(s,s) + \mu d(s_1,s_1)$$

$$d(s,s_1) \leq (\alpha+\beta+\gamma) d(s,s_1)$$

$\Rightarrow d(s,s_1) \leq 0$thus $s = s_1$.

Therefore f, g and h have a unique common fixed point.

Theorem 2.2. Let (X,d) be a metric space, and P be a normal cone with normal constant L. Suppose mappings $f,g,h : X \to X$ satisfy the contractive condition :

$$d(fx,gy) \leq \alpha d(hx,gy) + \beta[d(fx,fy) + d(fx,gy)] + \gamma[d(fx,hx) + d(gy,hy)] \tag{2.2.1}$$

Where $\alpha,\beta,\gamma > 0$and $0 \leq \alpha+\beta+\gamma < \frac{1}{2}$. If $f(X) \cup g(X) \subset h(X)$ is complete subspace of X. Then the maps f, g and h have a coincidence point $z$in X. Moreover if (f,h) and $(g,h)$are (IT)-Commutative at z,then f, g and h have unique common fixed point.

Proof. Let x_0 be an arbitrary point of X, and define the sequence $\{y_n\}$ in X such that

$y_{2n} = fx_{2n} = hx_{2n+1}$and

$y_{2n+1} = gx_{2n+1} = hx_{2n+2}$,

for all $n = 0,1,2, \ldots$.. by equation (1), we have

$$d(y_{2n},y_{2n+1}) = d(fx_{2n},gx_{2n+1}) \leq \alpha d(hx_{2n},gx_{2n+1}) + \beta[d(fx_{2n},fx_{2n+1}) + d(fx_{2n},gx_{2n+1})] + \gamma[d(fx_{2n},hx_{2n}) + d(gx_{2n+1},hx_{2n+1})]$$

$$= \alpha d(y_{2n-1},y_{2n+1}) + \beta[d(y_{2n},y_{2n+1}) + d(y_{2n},y_{2n+1})] + \gamma d(y_{2n},y_{2n-1}) + d(y_{2n+1},y_{2n})]$$

$$\leq \alpha[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})] + \beta[d(y_{2n},y_{2n+1}) + d(y_{2n},y_{2n+1})] + \gamma[d(y_{2n},y_{2n-1}) + d(y_{2n+1},y_{2n})]$$

$$(1-\alpha-2\beta-\gamma) d(y_{2n},y_{2n+1}) \leq (\alpha+\gamma) d(y_{2n},y_{2n-1})$$

$$d(y_{2n},y_{2n+1}) \leq \frac{\alpha+\gamma}{1-\alpha-2\beta-\gamma} d(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1}) \leq k d(y_{2n-1},y_{2n})$$

Where $k = \frac{\alpha+\gamma}{1-\alpha-2\beta-\gamma}$.

Similarly, it can be shown that

$$d(y_{2n+1},y_{2n+2}) \leq d(y_{2n},y_{2n+1})$$

Therefore, for all n,

$$d(y_{n+1},y_{n+2}) \leq k d(y_n,y_{n+1}) \leq \cdots \leq k^{n+1} d(y_0,y_1)$$

Now, for any $m > n$,

$$d(y_n,y_m) \leq d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + + d(y_{m-1},y_m) \leq [k^n + k^{n+1} + \cdots + k^{m-1}) d(y_1,y_0) \leq \frac{k^n}{1-k} d(y_1,y_0)$$

From (1.3), we have

$\| d(y_n,y_m) \| \leq \frac{k^n}{1-k} L \| d(y_1,y_0) \|$.

Which implies that $d(y_n,y_m) \to 0$ as

$n,m \to \infty$, (since $k < 1$). Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{hx_n\}$.

Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete, there exists z in $h(X)$ such that $hx_n \to z$as $n \to \infty$. Consequently, we can find z in X such that $h(z) = s$. We shall show that $hz = fz = gz$. Consider,

$$d(fz,gx_{2n+1}) \leq \alpha d(hz,gx_{2n+1}) + \beta[d(fz,fx_{2n+1}) + d(fz,gx_{2n+1})] + \gamma[d(fz,hz) + d(gx_{2n+1},hx_{2n+1})]$$

Letting $n \to \infty$, we get

$$d(fz,s) \leq \alpha d(s,s) + \beta[d(fz,s) + d(fz,s)] + \gamma[d(fz,s) + d(s,s)]$$

$d(fz,s) \leq (2\beta+\gamma) d(fz,s)$, a contradiction

Therefore

$$fz = s = hz. \tag{2.2.2}$$

similarly

$$d(gz,fx_{2n}) \leq \alpha d(hz,fx_{2n}) + \beta[d(gz,gx_{2n}) + d(gz,fx_{2n})] + \gamma[d(gz,hz) + d(fx_{2n},hx_{2n})]$$

$$d(gz,s) \leq \alpha d(s,s) + \beta[d(gz,s) + d(gz,s)] + \gamma[d(gz,s) + d(s,s)]$$

$d(gz,s) \leq (2\beta+\gamma) d(gz,s)$, a contradiction.

Therefore,

$$gz = s = hz. \tag{2.2.3}$$

From (2.2.2) and (2.2.3), it follows that

$hz = fz = gz = s$, z is a coincidence point f, g, h. (2.2.4)

Since $(f,h),(g,h)$ are (IT)-commuting at z.

$$d(ffz,fz) = d(ffz,gz) \leq \alpha d(hfz,gz) + \beta[d(ffz,fz) + d(ffz,gz)] + \gamma[d(ffz,hfz) + d(gz,hz)]$$

$$\leq \alpha d(fhz,gz) + 2\beta d(ffz,fz) + \gamma[d(ffz,fhz) + d(gz,hz)]$$

$$\leq \alpha d(fz,gz) + 2\beta d(ffz,fz) + \gamma[d(ffz,fz) + d(gz,hz)]$$

$$< (2\beta+\gamma) d(ffz,fz)$$

a contradiction, since $k < 1$ and $fz = hz$, which implies $ffz = fz$.

$fz = ffz = fhz = hfz,$

$$\Rightarrow ffz = hfz = fz = s. \qquad (2.2.5)$$

Therefore

$fz = s$is a common fixed point of g and h. (2.2.6)

Similarly, we get,

$$gz = ggz = ghz = hgz,$$
$$\Rightarrow ggz = hgz = gz = s. \qquad (2.2.7)$$

Therefore

$gz = fz = s$is a common fixed point of g and h. (2.2.8)

In view of (2.2.6) and (2.2.8), it follows that f, g and h have a common fixed point namely s. The uniqueness of the common fixed point of s follows equation (2.2.1). Indeed, let s_1 is another common fixed point of $f, g$and h. Consider,

$$\begin{aligned} d(s, s_1) &= d(fs, gs_1) \\ &\leq \alpha d(hs, gs_1) + \beta[d(fs, fs_1) \\ &\quad + d(fs, gs_1)] + \gamma[d(fs, hs) + d(gs_1, hs_1)] \\ &= \alpha d(s, s_1) + \beta[d(s, s_1) + d(s, s_1)] + \gamma[d(s, s) \\ &\quad + \mu d(s_1, s_1)] \end{aligned}$$

$$d(s, s_1) \leq (\alpha + 2\beta) d(s, s_1)$$

$\Rightarrow d(s, s_1) \leq 0$thus $s = s_1$.

Therefore f, g and h have a unique common fixed point.

Theorem 2.3. Let (X, d) be a metric space, and P be a normal cone with normal constant L. Suppose mappings $f, g, h : X \rightarrow X$ satisfy the contractive condition :

$$\begin{aligned} d(fx, gy) \leq \alpha[d(hx, gy) + d(fx, fy) + d(fy, gx)] \\ +\beta[d(fx, gy) + d(fx, hx) \\ +d(gy, hy)] \end{aligned} \qquad (2.3.1)$$

Where $\alpha, \beta > 0$and $4\alpha + 3\beta < 1$. If $f(X) \cup g(X) \subset h(X)$ is complete subspace of X. Then the maps f, g and h have a coincidence point $z$in X. Moreover if (f, h) and $(g, h)$are (IT)-Commutative at z,then f, g and h have unique common fixed point.

Proof. Let x_0 be an arbitrary point of X, and define the sequence $\{y_n\}$ in X such that

$y_{2n} = fx_{2n} = hx_{2n+1}$and

$y_{2n+1} = gx_{2n+1} = hx_{2n+2}$,

for all $n = 0, 1, 2, \ldots$.. by equation (1), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq \alpha[d(hx_{2n}, gx_{2n+1}) + d(fx_{2n}, fx_{2n+1}) \\ &\quad + d(fx_{2n+1}, gx_{2n})] + \beta[d(fx_{2n}, gx_{2n+1}) \\ &\quad + d(fx_{2n}, hx_{2n}) + d(gx_{2n+1}, hx_{2n+1})] \\ &= \alpha[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})] \\ &\quad + \beta[d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1}) \\ &\quad + d(y_{2n+1}, y_{2n})] \\ &\leq \alpha[\{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\} + 2d(y_{2n}, y_{2n+1})] \\ &\quad + \beta[2d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1})] \end{aligned}$$

$$(1 - 3\alpha - 2\beta) d(y_{2n}, y_{2n+1}) \leq (\alpha + \beta) d(y_{2n}, y_{2n-1})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{\alpha + \beta}{1 - 3\alpha - 2\beta} d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq k d(y_{2n-1}, y_{2n})$$

Where $k = \frac{\alpha+\beta}{1-3\alpha-2\beta}$.

Similarly, it can be shown that

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1})$$

Therefore, for all n,

$$d(y_{n+1}, y_{n+2}) \leq k d(y_n, y_{n+1}) \leq \cdots \leq k^{n+1} d(y_0, y_1)$$

Now, for any $m > n$,

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots \\ &\quad + d(y_{m-1}, y_m) \\ &\leq [k^n + k^{n+1} + \cdots + k^{m-1}) d(y_1, y_0) \\ &\leq \frac{k^n}{1 - k} d(y_1, y_0) \end{aligned}$$

From (1.3), we have

$\| d(y_n, y_m) \| \leq \frac{k^n}{1-k} L \| d(y_1, y_0) \|$.

Which implies that $d(y_n, y_m) \rightarrow 0$ as

$n, m \rightarrow \infty$, (since $k < 1$). Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{hx_n\}$.

Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete, there exists z in $h(X)$ such that $hx_n \rightarrow z$as $n \rightarrow \infty$. Consequently, we can find z in X such that $h(z) = s$. We shall show that $hz = fz = gz$. Consider,

$$\begin{aligned} d(fz, gx_{2n+1}) &\leq \alpha[d(hz, gx_{2n+1}) + d(fz, fx_{2n+1}) \\ &\quad + d(fx_{2n+1}, gz)] + \beta[d(fz, gx_{2n+1}) \\ &\quad + d(fz, hz) + d(gx_{2n+1}, hx_{2n+1})] \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned} d(fz, s) \leq \alpha[d(s, s) + d(fz, s) + d(s, s)] + \beta[d(fz, s) \\ + d(fz, s) + d(s, s)] \end{aligned}$$

$d(fz, s) \leq (\alpha + 2\beta) d(fz, s)$, a contradiction

Therefore

$$fz = s = hz. \qquad (2.3.2)$$

similarly

$$\begin{aligned} d(gz, fx_{2n}) &\leq \alpha[d(hz, fx_{2n}) + d(gz, gx_{2n}) + d(fx_{2n}, gz)] \\ &\quad + \beta[d(gz, fx_{2n}) + d(gz, hz) \\ &\quad + d(fx_{2n}, hx_{2n})] \end{aligned}$$

$$\begin{aligned} d(gz, s) \leq \alpha[d(s, s) + d(gz, s) + d(s, gz)] + \beta[d(gz, s) \\ + d(gz, s) + d(s, s)] \end{aligned}$$

$d(gz,s) \leq (2\alpha + 2\beta)d(gz,s)$, a contradiction.

Therefore,

$$gz = s = hz. \tag{2.3.3}$$

From (2.3.2) and (2.3.3), it follows that

$hz = fz = gz = s$, z is a coincidence point f,g,h. (2.3.4)

Since $(f,h),(g,h)$ are (IT)-commuting at z.

$$\begin{aligned} d(ffz,fz) &= d(ffz,gz) \\ &\leq \alpha[d(hfz,gz) + d(ffz,fz) \\ &\quad + d(fgz,gfz)] + \beta[d(ffz,gz) \\ &\quad + d(ffz,hfz) + d(gz,hz)] \\ &\leq \alpha[d(fhz,gz) + d(ffz,fz) + d(fgz,fgz)] \\ &\quad + \beta[d(ffz,gz) + d(ffz,fhz) \\ &\quad + d(gz,hz)] \\ &\leq \alpha[d(fz,gz) + d(ffz,fz) + d(fz,fz)] + \beta[d(ffz,gz) \\ &\quad + d(ffz,fz) + d(gz,hz)] \\ &< (\alpha + 2\beta)d(ffz,fz) \end{aligned}$$

a contradiction, since $k < 1$ and $fz = hz$, which implies $ffz = fz$.

$$fz = ffz = fhz = hfz,$$
$$\Rightarrow ffz = hfz = fz = s. \tag{2.3.5}$$

Therefore

$fz = s$is a common fixed point of g and h. (2.3.6)

Similarly, we get,

$$gz = ggz = ghz = hgz,$$
$$\Rightarrow ggz = hgz = gz = s. \tag{2.3.7}$$

Therefore

$gz = fz = s$is a common fixed point of g and h. (2.3.8)

In view of (2.3.6) and (2.3.8), it follows that f,g and h have a common fixed point namely s. The uniqueness of the common fixed point of s follows equation (2.3.1). Indeed, let s_1 is another common fixed point of $f,g$and h. Consider,

$$\begin{aligned} d(s,s_1) &= d(fs,gs_1) \\ &\leq \alpha[d(hs,gs_1) + d(fs,fs_1) + d(fs_1,gs)] \\ &\quad + \beta[d(fs,gs_1) + d(fs,hs) + d(gs_1,hs_1)] \\ &= \alpha[d(s,s_1) + d(s,s_1) + d(s_1,s)] + \beta[d(s,s_1) + d(s,s) \\ &\quad + d(s_1,s_1)] \end{aligned}$$

$$d(s,s_1) \leq (3\alpha + \beta)d(s,s_1)$$

$\Rightarrow d(s,s_1) \leq 0$thus $s = s_1$.

Therefore f,g and h have a unique common fixed point.

Theorem 2.4. Let (X,d) be a metric space, and P be a normal cone with normal constant L. Suppose mappings $f,g,h : X \to X$ satisfy the contractive condition :

$$d(fx,gy) \leq \alpha[d(hx,gy) + d(fx,fy)]$$

$$+\beta[d(fy,gx) + d(fx,gy)]$$

$$+\gamma[d(fx,hx)$$

$+d(gy,hy)]$ (2.4.1)

Where $\alpha,\beta,\gamma > 0$ and $3\alpha + 2\beta + 2\gamma < 1$. If $f(X) \cup g(X) \subset h(X)$ is complete subspace of X. Then the maps f,g and h have a coincidence point $z$in X. Moreover if (f,h) and (g,h) are (IT)-Commutative at z,then f,g and h have unique common fixed point.

Proof. Let x_0 be an arbitrary point of X, and define the sequence $\{y_n\}$ in X such that

$y_{2n} = fx_{2n} = hx_{2n+1}$and

$y_{2n+1} = gx_{2n+1} = hx_{2n+2}$,

for all $n = 0,1,2,\ldots..$ by equation (1), we have

$$\begin{aligned} d(y_{2n},y_{2n+1}) &= d(fx_{2n},gx_{2n+1}) \\ &\leq \alpha[d(hx_{2n},gx_{2n+1}) + d(fx_{2n},fx_{2n+1})] \\ &\quad + \beta[d(fx_{2n+1},gx_{2n}) + d(fx_{2n},gx_{2n+1})] \\ &\quad + \gamma[d(fx_{2n},hx_{2n}) + d(gx_{2n+1},hx_{2n+1})] \end{aligned}$$

$$\begin{aligned} &= \alpha[d(y_{2n-1},y_{2n+1}) + d(y_{2n},y_{2n+1})] + \beta[d(y_{2n+1},y_{2n}) \\ &\quad + d(y_{2n},y_{2n+1})] + \gamma[d(y_{2n},y_{2n-1}) \\ &\quad + d(y_{2n+1},y_{2n})] \end{aligned}$$

$$\begin{aligned} &\leq \alpha[\{d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})\} + d(y_{2n},y_{2n+1})] \\ &\quad + 2\beta d(y_{2n},y_{2n+1}) + \gamma[d(y_{2n},y_{2n-1}) \\ &\quad + d(y_{2n},y_{2n+1})] \end{aligned}$$

$$(1 - 2\alpha - 2\beta - \gamma)d(y_{2n},y_{2n+1}) \leq (\alpha + \gamma)d(y_{2n},y_{2n-1})$$

$$d(y_{2n},y_{2n+1}) \leq \frac{\alpha + \gamma}{1 - 2\alpha - 2\beta - \gamma} d(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1}) \leq kd(y_{2n-1},y_{2n})$$

Where $k = \frac{\alpha+\gamma}{1-2\alpha-2\beta-\gamma}$.

Similarly, it can be shown that

$$d(y_{2n+1},y_{2n+2}) \leq d(y_{2n},y_{2n+1})$$

Therefore, for all n,

$$d(y_{n+1},y_{n+2}) \leq kd(y_n,y_{n+1}) \leq \cdots \leq k^{n+1}d(y_0,y_1)$$

Now, for any $m > n$,

$$\begin{aligned} d(y_n,y_m) &\leq d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \cdots \\ &\quad + d(y_{m-1},y_m) \\ &\leq [k^n + k^{n+1} + \cdots + k^{m-1})d(y_1,y_0) \\ &\leq \frac{k^n}{1-k} d(y_1,y_0) \end{aligned}$$

From (1.3), we have

$\| d(y_n,y_m) \| \leq \frac{k^n}{1-k} L \| d(y_1,y_0) \|$.

Which implies that $d(y_n,y_m) \to 0$ as

$n,m \to \infty$, (since $k < 1$). Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{hx_n\}$.

Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete, there exists z in $h(X)$ such that $hx_n \to z$as $n \to \infty$.

Consequently, we can find z in X such that $h(z) = s$. We shall show that $hz = fz = gz$. Consider,

$$\begin{aligned} d(fz, gx_{2n+1}) \leq \alpha[d(hz, gx_{2n+1}) + d(fz, fx_{2n+1})] \\ + \beta[d(fx_{2n+1}, gz) + d(fz, gx_{2n+1})] \\ + \gamma[d(fz, hz) + d(gx_{2n+1}, hx_{2n+1})] \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(fz, s) \leq \alpha[d(s, s) + d(fz, s)] + \beta[d(s, s) + d(fz, s)] \\ + \gamma[d(fz, s) + d(s, s)] \end{aligned}$$

$d(fz, s) \leq (\alpha + \beta + \gamma)d(fz, s)$, a contradiction

Therefore

$$fz = s = hz. \quad (2.4.2)$$

similarly

$$\begin{aligned} d(gz, fx_{2n}) \leq \alpha[d(hz, fx_{2n}) + d(gz, gx_{2n})] \\ + \beta[d(fx_{2n}, gz) + d(gz, fx_{2n})] \\ + \gamma[d(gz, hz) + d(fx_{2n}, hx_{2n})] \end{aligned}$$

$$\begin{aligned} d(gz, s) \leq \alpha[d(s, s) + d(gz, s)] + \beta[d(s, gz) + d(gz, s)] \\ + \gamma[d(gz, s) + d(s, s)] \end{aligned}$$

$d(gz, s) \leq (\alpha + 2\beta + \gamma)d(gz, s)$, a contradiction.

Therefore,

$$gz = s = hz. \quad (2.4.3)$$

From (2.4.2) and (2.4.3), it follows that

$hz = fz = gz = s$, z is a coincidence point f, g, h. (2.4.4)

Since $(f, h), (g, h)$ are (IT)-commuting at z.

$$\begin{aligned} d(ffz, fz) &= d(ffz, gz) \\ &\leq \alpha[d(hfz, gz) + d(ffz, fz)] \\ &\quad + \beta[d(fgz, gfz) + d(ffz, gz)] \\ &\quad + \gamma[d(ffz, hfz) + d(gz, hz)] \\ &\leq \alpha[d(fhz, gz) + d(ffz, fz)] + \beta[d(fgz, fgz) \\ &\quad + d(ffz, gz)] + \gamma[d(ffz, fhz) \\ &\quad + d(gz, hz)] \\ &\leq \alpha[d(fz, gz) + d(ffz, fz)] + \beta[d(fz, fz) + d(ffz, gz)] \\ &\quad + \gamma[d(ffz, fz) + d(gz, hz)] \\ &< (\alpha + \beta + \gamma)d(ffz, fz) \end{aligned}$$

a contradiction, since $k < 1$ and $fz = hz$, which implies $ffz = fz$.

$$\begin{aligned} &fz = ffz = fhz = hfz, \\ &\Rightarrow ffz = hfz = fz = s. \end{aligned} \quad (2.4.5)$$

Therefore

$fz = s$is a common fixed point of g and h. (2.4.6)

Similarly, we get,

$$\begin{aligned} &gz = ggz = ghz = hgz, \\ &\Rightarrow ggz = hgz = gz = s. \end{aligned} \quad (2.4.7)$$

Therefore

$gz = fz = s$is a common fixed point of g and h. (2.4.8)

In view of (2.4.6) and (2.4.8), it follows that f, g and h have a common fixed point namely s. The uniqueness of the common fixed point of s follows equation (2.4.1). Indeed, let s_1 is another common fixed point of $f, g$and h. Consider,

$$\begin{aligned} d(s, s_1) &= d(fs, gs_1) \\ &\leq \alpha[d(hs, gs_1) + d(fs, fs_1)] \\ &\quad + \beta[d(fs_1, gs) + d(fs, gs_1)] \\ &\quad + \gamma[d(fs, hs) + d(gs_1, hs_1)] \\ &= \alpha[d(s, s_1) + d(s, s_1)] + \beta[d(s_1, s) + d(s, s_1)] \\ &\quad + \gamma[d(s, s) + d(s_1, s_1)] \\ d(s, s_1) &\leq (2\alpha + 2\beta)d(s, s_1) \end{aligned}$$

$\Rightarrow d(s, s_1) \leq 0$thus $s = s_1$.

Therefore f, g and h have a unique common fixed point.

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